Convex Optimization for Fun and Profit

Optimally routing through DEXs

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Based on work by G. Angeris, T. Chitra, A. Evans, and S. Boyd

mev.day, April 22, 2022

Why care about convexity?

Routing (arbitrage, swaps, etc.) is a convex[1] optimization problem, so it can be efficiently solved to global optimality.

[1] when we ignore gas

Outline

Review: Constant Function Market Makers

- Most DEXs are implemented as *constant function market makers* (CFMMs)
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- Is concave and increasing
- Accepts trade $\Delta \to \Lambda$ if $\varphi(R + \gamma\Delta - \Lambda) \geq \varphi(R)$.

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- Geometric mean trading function (Balancer, Uniswap, etc...):

$$\varphi(R) = \prod_{i=1}^{n} R_i^{w_i}$$

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- Curve:

$$\varphi(R) = 1^T R - \alpha \prod_{i=1}^{n} R_i^{-1}$$

  where $\alpha > 0$.

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Review: Constant Function Market Makers

Formalizing Routing

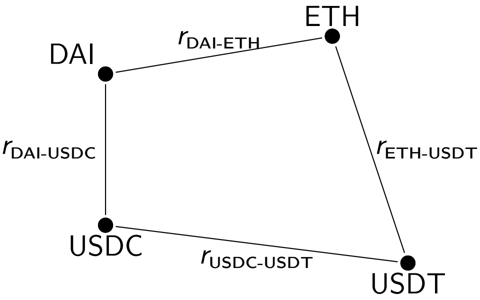
Swaps, Arbitrage, and Some of My Favorite Things

When in Doubt, Take the Dual

Wrap Up

Networks of CFMMs

- Common representation: undirected graph with exchange rates



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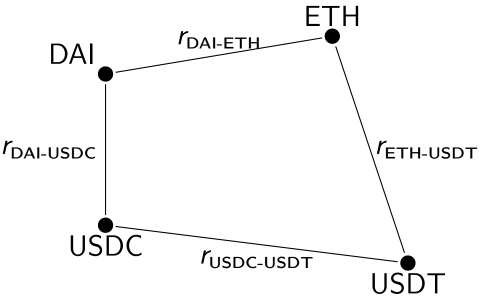

- But how to handle three pools? Multiple CFMMs?

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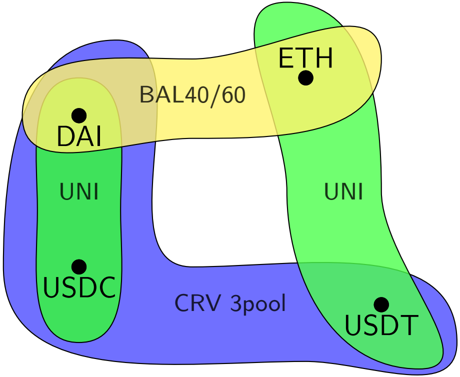

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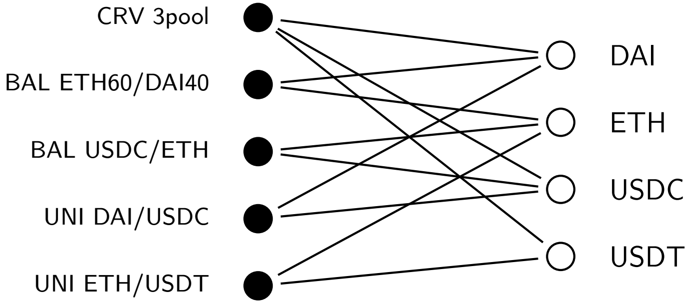

- Good bookkeeping is essential!

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- Trade accepted if $\varphi_i(R_i + \gamma_i \Delta_i - \Lambda_i) \geq \varphi_i(R_i)$

Networks of CFMMs

- Matrices A_i map token's *local* index in CFMM i to global index, *e.g.*,

Token	Local Index	Global Index
DAI	1	3
ETH	2	1

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- The overall net trade with the network is

$$\Psi = \sum_{i=1}^{m} A_i(\Lambda_i - \Delta_i)$$

Simplifying the Model

- We ignore gas fees
- We don't worry about transaction execution ordering
- We can return to these later...

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- Each individual CFMM is defined by trading constraints

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- Utility function U gives our satisfaction with the net trade
- We can also use U to encode constraints
- **Arbitrage:** Find the best entirely nonnegative net trade

$$U(\Psi) = c^T\Psi - \mathbb{I}(\Psi \geq 0)$$

  - The vector c is a positive price vector
  - Indicator function $\mathbb{I}(\Psi \geq 0) = 0$ if $\Psi \geq 0$ and $+\infty$ otherwise

Swaps: trade token i for j

- Goal: maximize output of token j given fixed input of token i
- Constraints: input exactly Δ^i of token i and only get token j

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- More generally, we can optimally purchase or liquidate a basket of tokens
- Capturing "arbitrage" opportunities as part of the swap

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- The dual problem: finding the optimal prices
- Idea: your utility function induces personal "shadow" prices (marginal utilities) at which you value each token
- Given these prices, you can arbitrage each CFMM independently & in parallel
- Strong duality $\implies$ dual problem has the same optimal value

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- To add a DEX, **only need to define this arbitrage function**

Check out `CFMMRouter.jl` & the docs.

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- This means it can be solved quickly to global optimality
- We construct an efficient algorithm using convex duality
- This algorithm is implemented in `CFMMRouter.jl`

Future work includes expanding this framework

- Routing with gas fees (nonconvex—need good heuristics)
- Routing through liquidations
- Routing with probabilistic constraints (when TXs may fail)
- Additional features in `CFMMRouter.jl`

Thank you!

- **Paper:** "Optimal routing for constant function market makers"
- **Package:** `CFMMRouter.jl`
- **Contact:** `@theo_diamandis`

Appendix

Routing with gas fees

- Gas cost for CFMM i is q_i
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- ...but we have good heuristics for this type of problem

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- Use ℓ_1 norm to approximate cardinality of trade vectors Δ_i
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- **Answer 1:** if solving the dual, only need to define $\mathrm{arb}(\cdot)$
- This is relatively easy: closed form solution for each tick
- **Answer 2:** The φ constraint is a bit of a lie...
- Only need a convex reachable reserve set:

$$\varphi(R + \gamma\Delta - \Lambda) \geq \varphi(R) \iff R + \gamma\Delta - \Lambda \in S(R)$$